

Polyhedral fans

Polyhedral fans play an important role in *toric geometry*, the theory of polytopes and τ -tilting theory. A *polyhedral cone* in \mathbb{R}^n is the non-negative linear span of linearly independent vectors. A *fan* Σ in \mathbb{R}^n is a collection of polyhedral cones in \mathbb{R}^n satisfying the following:

1. Each face of a cone in Σ is also contained in Σ .
2. The intersection of two cones in Σ is a face of each of the two cones.

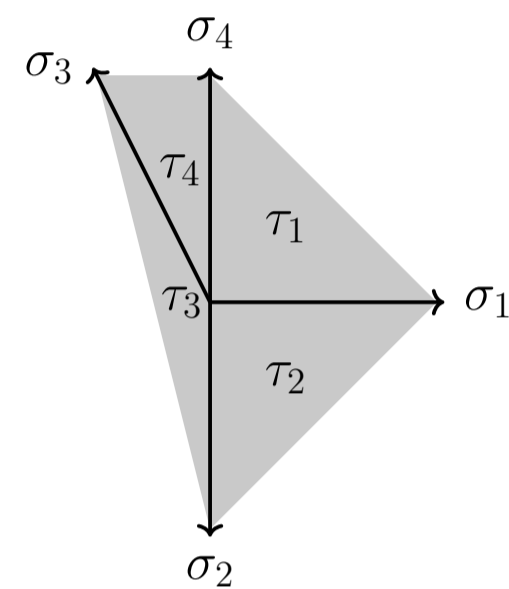


Figure 1. Fan $\Sigma_{\mathbb{F}_a}$ of a Hirzebruch surface \mathbb{F}_a

For a cone $\sigma \in \Sigma$, define $\text{star}(\sigma) = \{\tau \in \Sigma : \sigma \subseteq \tau\}$ and the orthogonal projection

$$\pi_\sigma : \mathbb{R}^n \rightarrow \text{span}\{\sigma\}^\perp.$$

An admissible partition of the fan

Potential identifications of a cone $\sigma \in \Sigma$:

$$\mathcal{E}_\sigma := \{\kappa \in \Sigma : \text{span}\{\sigma\}^\perp = \text{span}\{\kappa\}^\perp \text{ and } \pi_\sigma(\text{star}(\sigma)) = \pi_\kappa(\text{star}(\kappa))\}.$$

Partitioning the sets \mathcal{E}_σ into *actual identifications* \rightarrow Partition \mathfrak{P} of Σ . Such a partition \mathfrak{P} is called *admissible* if whenever $\sigma_1 \sim \sigma_2$ are such that $\pi_{\sigma_1}(\tau_1) = \pi_{\sigma_2}(\tau_2)$ for some $\tau_1 \in \text{star}(\sigma_1)$ and $\tau_2 \in \text{star}(\sigma_2)$, then $\tau_1 \sim \tau_2$.

Lemma. Admissible partitions exist.

The associated category

Given a fan Σ and an admissible partition of its cones \mathfrak{P} , define the *category of the partitioned fan*, denoted by $\mathcal{C}(\Sigma, \mathfrak{P})$, as follows:

1. Objects of $\mathcal{C}(\Sigma, \mathfrak{P})$ are equivalence classes $[\sigma]$ of the partition \mathfrak{P} .
2. $\text{Hom}_{\mathcal{C}(\Sigma, \mathfrak{P})}([\sigma], [\tau])$ consists of equivalence classes of objects in

$$\bigcup_{\sigma' \in [\sigma], \tau' \in [\tau]} \text{Hom}_\Sigma(\sigma', \tau')$$

under the relation $f_{\sigma_1 \tau_1} \sim f_{\sigma_2 \tau_2}$ if and only if $\pi_{\sigma_1}(\tau_1) = \pi_{\sigma_2}(\tau_2)$.

3. Given $[f_{\sigma\tau}] \in \text{Hom}_{\mathcal{C}(\Sigma, \mathfrak{P})}([\sigma], [\tau])$ and $[f_{\tau\kappa}] \in \text{Hom}_{\mathcal{C}(\Sigma, \mathfrak{P})}([\tau], [\kappa])$, their composition is defined as $[f_{\tau\kappa}] \circ [f_{\sigma\tau}] = [f_{\sigma\kappa}]$.

Lemma. Composition is well-defined (cf. [4, Lem. 3.9, 3.10]).

Cubical categories

The *standard k -cube category* \mathcal{J}^k is the poset category on subsets of $\{1, \dots, k\}$ under inclusion. For any morphism $(A \xrightarrow{f} B)$ in some category \mathcal{C} , the *factorisation category* $\text{Fa}(\mathcal{C}, f)$ is the category whose objects are factorisations $A \xrightarrow{g} C \xrightarrow{h} B$ such that $h \circ g = f$ and whose morphisms are morphisms $\phi : C_1 \rightarrow C_2$ such that $\phi \circ g_1 = g_2$ and $h_1 = h_2 \circ \phi$. Given an object $(A \xrightarrow{g} C \xrightarrow{h} B)$ in $\text{Fa}(\mathcal{C}, f)$, we call g a *first factor of f* if g is irreducible in \mathcal{C} and h a *last factor of f* if h is irreducible in \mathcal{C} .

A *cubical category*, introduced by Igusa [3], is a small category \mathcal{C} with the following properties:

1. Every morphism $f : A \rightarrow B$ has a *rank* $\text{rk}(f)$ which is a non-negative integer such that $\text{rk}(g \circ f) = \text{rk}(f) + \text{rk}(g)$.
2. If $\text{rk}(f) = k$ then the factorisation category of f is isomorphic to the standard k -cube category, $\text{Fa}(\mathcal{C}, f) \cong \mathcal{J}^k$.
3. The forgetful functor $\text{Fa}(\mathcal{C}, f) \rightarrow \mathcal{C}$ taking $A \xrightarrow{g} C \xrightarrow{h} B$ to C is an embedding.
4. Every morphism of rank k is determined by its k first factors.
5. Every morphism of rank k is determined by its k last factors.

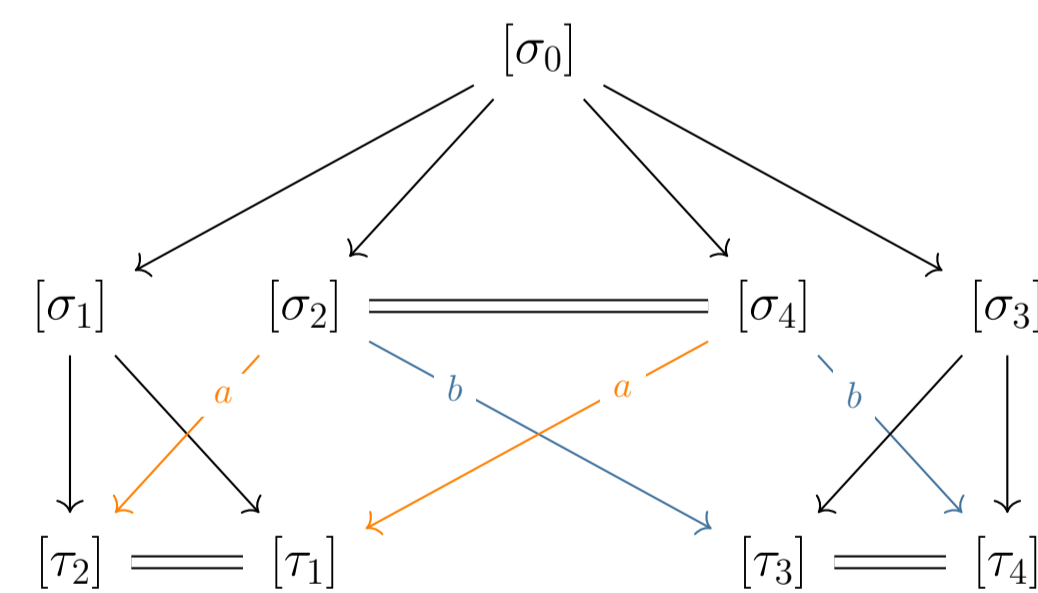
Theorem. The category $\mathcal{C}(\Sigma, \mathfrak{P})$ of a fan with an admissible partition is cubical.

Remark. The classifying space of a cubical category is a cube complex [3].

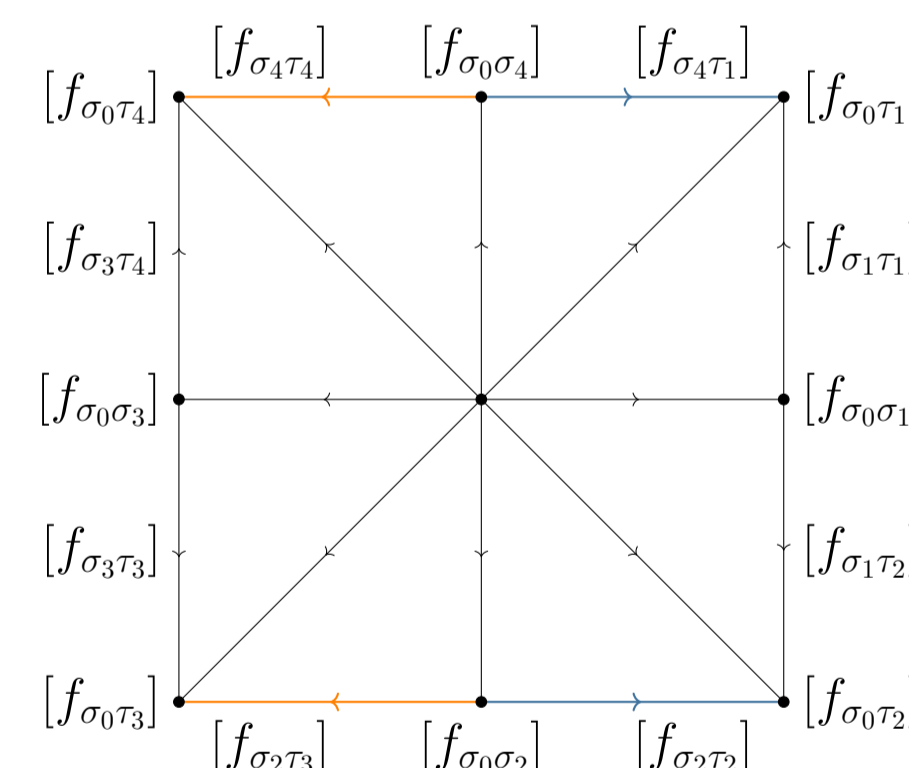
An example

Consider the fan $\Sigma_{\mathbb{F}_a}$ from Figure 1. with the identification $\sigma_2 \sim \sigma_4$. Because $\pi_{\sigma_2}(\tau_2) = \pi_{\sigma_4}(\tau_1)$ and $\pi_{\sigma_2}(\tau_3) = \pi_{\sigma_4}(\tau_4)$ we must also identify $\tau_1 \sim \tau_2$ and $\tau_3 \sim \tau_4$ to make the partition admissible. We obtain the following partition:

$$\mathfrak{P} = \{\{\sigma_0\}, \{\sigma_1\}, \{\sigma_3\}, \{\sigma_2, \sigma_4\}, \{\tau_1, \tau_2\}, \{\tau_3, \tau_4\}\}.$$



(a) The category $\mathcal{C}(\Sigma_{\mathbb{F}_a}, \mathfrak{P})$



(b) The 2-cell of the CW-complex $\mathcal{B}\mathcal{C}(\Sigma_{\mathbb{F}_a}, \mathfrak{P})$

Remark. We may also identify all rank 2 cones.

Classifying space

The *classifying space* $\mathcal{B}\mathcal{C}$ of a category \mathcal{C} is the geometric realisation of the *simplicial nerve* of the category. 0-simplices correspond to the objects of \mathcal{C} and the non-degenerate k -simplices correspond to the chains of composable non-identity morphisms $(X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_k} X_k)$ in \mathcal{C} .

Theorem. The classifying space $\mathcal{B}\mathcal{C}(\Sigma, \mathfrak{P})$ is a n -dimensional CW-complex having one cell $e([\sigma])$ of dimension $k = n - \dim(\sigma)$ for each equivalence class $[\sigma] \in \mathfrak{P}$. The k -cell $e([\sigma])$ is the union of the factorisation cubes of the morphisms $[f_{\sigma\tau}]$, where τ is an n -dimensional cone in $\text{star}(\sigma)$.

g -fans in τ -tilting theory

Let $M \in \text{mod } A$. Let $P_1 \rightarrow P_0 \rightarrow T \rightarrow 0$ be a minimal projective presentation and write $P_0 = \bigoplus_{i=1}^n P(i)^{a_i}$ and $P_1 = \bigoplus_{i=1}^n P(i)^{b_i}$. Then the *g -vector* of M is

$$g^M := (a_1 - b_1, a_2 - b_2, \dots, a_n - b_n).$$

Theorem. Write $\bar{\mathcal{C}}_{(T,P)} := \text{cone}\{g^{T_1}, \dots, g^{T_k}, -g^{P_{k+1}}, \dots, -g^{P_l}\}$ for the cone of g -vectors of the indecomposable direct summands of a τ -rigid pair (T, P) . Then

$$\Sigma(A) := \{\bar{\mathcal{C}}_{(T,P)} : (T, P) \in \text{s}\tau\text{-rigid } A\}$$

is a polyhedral fan, called the *g -fan*. See also [1].

Remark. The τ -cluster morphism category $\mathcal{C}(A)$ is $\mathcal{C}(\Sigma(A), \mathfrak{P})$ for some \mathfrak{P} .

Theorem. For any τ -tilting finite algebra A , there exists a faithful group functor $\mathcal{C}(A) \rightarrow G(A)$ to the picture group (with the usual poset on maximal cones).

Corollary. Let A be a rank 2 algebra. The classifying space of the τ -cluster morphism $\mathcal{C}(A)$ category is a $K(G(A), 1)$.

Picture group

A *fan poset* is a pair (Σ, \mathcal{P}) where Σ is a finite complete fan in \mathbb{R}^n and \mathcal{P} is a poset on Σ^n such that (1) for every interval I of \mathcal{P} , the union of all maximal cones in I is a polyhedral cone and (2) for every cone $\sigma \in \Sigma$, the set of maximal cones containing σ is an interval in \mathcal{P} , which we denote by $[\sigma^-, \sigma^+]$.

Definition Let $(\Sigma, \mathfrak{P}, \mathcal{P})$ be a partitioned fan poset. The *picture group* $G(\Sigma, \mathfrak{P}, \mathcal{P})$ has generators $\{X_{[\sigma]} : \sigma \in \Sigma^{n-1}\}$ and the following sets of relations:

- $X_{[\sigma_1]} \dots X_{[\sigma_k]} = X_{[\sigma'_1]} \dots X_{[\sigma'_k]}$, whenever $(\sigma_1, \dots, \sigma_k)$ and $(\sigma'_1, \dots, \sigma'_k)$ are two distinct sequences of codimension 1 cones labelling the arrows of some interval $[\tau_1, \tau_2]$.
- $X_{[\sigma_1^-, \tau_1^-]} = X_{[\sigma_2^-, \tau_2^-]}$, which correspond to the products of generators labelling paths of the two intervals $[\sigma_1^-, \tau_1^-]$ and $[\sigma_2^-, \tau_2^-]$ whenever $[f_{\sigma_1 \tau_1}] = [f_{\sigma_2 \tau_2}]$ in $\mathcal{C}(\Sigma, \mathfrak{P})$.

Proposition. The τ -cluster morphism category of the g -fan with the usual ordering on maximal cones given by $(\text{tors } A, \subseteq)$ satisfies this.

Theorem. The functor $\mathcal{C}(\Sigma, \mathfrak{P}) \rightarrow G(\Sigma, \mathfrak{P}, \mathcal{P})$ sending $[f_{\sigma\tau}] \mapsto [\sigma^-, \tau^-]$ is faithful when Σ is a rank 2 fan and \mathcal{P} does not annihilate any generators. Then $\mathcal{B}\mathcal{C}(\Sigma, \mathfrak{P})$ is a $K(G(\Sigma, \mathfrak{P}, \mathcal{P}), 1)$.

Lattice of admissible partitions

Let \mathfrak{P}_1 and \mathfrak{P}_2 be partitions of a fan Σ . We say that \mathfrak{P}_1 is a *finer* partition than \mathfrak{P}_2 if

$$\sigma \sim_{\mathfrak{P}_1} \tau \implies \sigma \sim_{\mathfrak{P}_2} \tau$$

for $\sigma, \tau \in \Sigma$. In this case, we write $\mathfrak{P}_1 \leq \mathfrak{P}_2$ and say \mathfrak{P}_2 is *coarser* than \mathfrak{P}_1 . Denote by $\text{APart}(\Sigma)$ the set of all admissible partitions of a fan Σ .

Proposition. The partially ordered set $\text{APart}(\Sigma)$ is a complete lattice.

References

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- [4] S. Schroll, A. Tattar, H. Treffinger, and N. Williams. A geometric perspective on the τ -cluster morphism category, 2023. [arXiv:2302.12217](https://arxiv.org/abs/2302.12217).